

Unitary representations of the quantum algebra $\text{su}_q(2)$ on a real two-dimensional sphere for $q \in \mathbb{R}^+$ or generic $q \in S^1$

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Abstract

Some time ago, Rideau and Winternitz introduced a realization of the quantum algebra $\text{su}_q(2)$ on a real two-dimensional sphere, or a real plane, and constructed a basis for its representations in terms of q -special functions, which can be expressed in terms of q -Vilenkin functions, and are related to little q -Jacobi functions, q -spherical functions, and q -Legendre polynomials. In their study, the values of q were implicitly restricted to $q \in \mathbb{R}^+$. In the present paper, we extend their work to the case of generic values of $q \in S^1$ (i.e., q values different from a root of unity). In addition, we unitarize the representations for both types of q values, $q \in \mathbb{R}^+$ and generic $q \in S^1$, by determining some appropriate scalar products. From the latter, we deduce the orthonormality relations satisfied by the q -Vilenkin functions.

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I INTRODUCTION

As is well known, most special functions of mathematical physics admit extensions to a base q , which are called q -special functions [1, 2, 3]. In the same way as Lie algebras and their representations provide a unifying framework for the former, quantum algebras [4] are relevant to the study of the latter (see e.g. [5] and references quoted therein).

Some time ago, Rideau and Winternitz [6] introduced a realization of the quantum algebra $\mathfrak{su}_q(2)$ on a real sphere S^2 (or, via a stereographic projection, on a real plane), and constructed a basis for its irreducible representations (irreps) in terms of some functions $\Psi_{MNq}^J(\theta, \phi) \propto P_{MNq}^J(\cos \theta) \exp(-i(M+N)\phi)$. The functions $P_{MNq}^J(\cos \theta)$ were called q -Vilenkin functions because, for $q = 1$, they reduce to functions $P_{MN}^J(\cos \theta)$ introduced by Vilenkin [7, 8], and related to Jacobi polynomials.

Rideau and Winternitz did establish various interesting results for the q -Vilenkin functions, including their recursion relations, explicit expression, generating function, and symmetry relations. They also compared them with other q -special functions, such as q -hypergeometric series, little q -Jacobi functions, q -spherical functions, and q -Legendre polynomials. Recently, the latter polynomials were further studied by Schmidt along similar lines [9].

The realization of $\mathfrak{su}_q(2)$ on S^2 , introduced by Rideau and Winternitz, was used by one of the present authors (MIA) to set up $\mathfrak{su}_q(2)$ -invariant Schrödinger equations in the usual framework of quantum mechanics [10]. The corresponding radial equations can be easily solved for the “free” $\mathfrak{su}_q(2)$ -invariant particle [10], as well as for the Coulomb [10] and oscillator [11] potentials.

Although not explicitly stated in Ref. [6], the values of the deformation parameter q , considered there, are restricted to $q \in \mathbb{R}^+$. Close examination indeed shows that the explicit form of the function $Q_{Jq}(\eta)$, $\eta \equiv \cot^2(\theta/2)$, entering the definition of the q -Vilenkin functions [6], is not valid for half-integer J values, whenever q runs over the unit circle.

Though important both from the q -special function viewpoint, and from that of their applications in quantum mechanics, the question of the $\mathfrak{su}_q(2)$ irrep unitarity was also left unsolved by Rideau and Winternitz. They only noticed [6] that their realization of $\mathfrak{su}_q(2)$

on S^2 is not unitary with respect to the scalar product used to unitarize the corresponding realization of $\mathfrak{su}(2)$, and that a new scalar product should therefore be determined to cope with this drawback.

The purpose of the present paper is twofold: firstly, to find a solution for $Q_{Jq}(\eta)$ for generic $q \in S^1$ (i.e., for q different from a root of unity), and secondly, to unitarize the representations for both $q \in \mathbb{R}^+$, and generic $q \in S^1$. As a consequence, the explicit orthonormality relations of the q -Vilenkin and related functions will be established.

In Sec. II, the representations of $\mathfrak{su}_q(2)$ on S^2 , derived by Rideau and Winternitz, are briefly reviewed. The function $Q_{Jq}(\eta)$ is determined in Sec. III. The unitarization of the representations is dealt with in Sec. IV. Sec. V contains the conclusion.

II REPRESENTATIONS OF $\mathfrak{su}_q(2)$ ON S^2

Let us consider functions $f(\theta, \phi)$ on a sphere S^2 , defined by $x_0^2 + y_0^2 + z_0^2 = 1/4$. These functions can also be viewed as functions on a real plane, via the stereographic projection $x = x_0/(1/2 - z_0)$, $y = y_0/(1/2 - z_0)$. In terms of spherical coordinates on S^2 and polar ones on the plane, we have

$$\begin{aligned} x_0 &= \frac{1}{2} \sin \theta \cos \phi, & y_0 &= \frac{1}{2} \sin \theta \sin \phi, & z_0 &= \frac{1}{2} \cos \theta, \\ x &= \rho \cos \phi, & y &= \rho \sin \phi, & \rho &= \cot \frac{\theta}{2}, \\ 0 &\leq \theta \leq \pi, & 0 &\leq \phi < 2\pi, & 0 &\leq \rho < \infty. \end{aligned} \tag{2.1}$$

Instead of the real variables x and y , one can use complex ones

$$z = x + iy = \rho e^{i\phi}, \quad \bar{z} = x - iy = \rho e^{-i\phi}. \tag{2.2}$$

Functions $f(\theta, \phi)$ on S^2 can thus be projected onto functions $f(\rho, \phi)$ on the real plane, or functions $f(z, \bar{z})$ of a complex variable and its conjugate.

The $\mathfrak{su}_q(2)$ generators H_3, H_+, H_- satisfy the commutation relations [4]

$$[H_3, H_{\pm}] = \pm H_{\pm}, \quad [H_+, H_-] = [2H_3]_q \equiv \frac{q^{2H_3} - q^{-2H_3}}{q - q^{-1}}, \tag{2.3}$$

and the Hermiticity properties

$$H_3^\dagger = H_3, \quad H_\pm^\dagger = H_\mp, \quad (2.4)$$

where in Eq. (2.3), we assume $q = e^\tau \in \mathbb{R}^+$, or $q = e^{i\tau} \in S^1$ (but different from a root of unity). From H_3 and H_\pm , one can construct a Casimir operator

$$\mathcal{C} = H_+ H_- + [H_3]_q [H_3 - 1]_q = H_- H_+ + [H_3]_q [H_3 + 1]_q, \quad (2.5)$$

such that $[\mathcal{C}, H_3] = [\mathcal{C}, H_\pm] = 0$.

The generators H_3 , H_+ , H_- can be realized [6] by the following operators, acting on functions $f(z, \bar{z})$ or $f(\theta, \phi)$,

$$\begin{aligned} H_3 &= -z\partial_z + \bar{z}\partial_{\bar{z}} - N = i\partial_\phi - N, \\ H_+ &= -z^{-1} [T]_q q^{\bar{T}-(N/2)} - q^{T+(N/2)} \bar{z} [\bar{T} - N]_q, \\ H_- &= z [T + N]_q q^{\bar{T}-(N/2)} + q^{T+(N/2)} \bar{z}^{-1} [\bar{T}]_q, \end{aligned} \quad (2.6)$$

where

$$T = z\partial_z = -\frac{1}{2}(\sin\theta\partial_\theta + i\partial_\phi), \quad \bar{T} = \bar{z}\partial_{\bar{z}} = -\frac{1}{2}(\sin\theta\partial_\theta - i\partial_\phi). \quad (2.7)$$

For future use, it is also convenient to write H_\pm in terms of polar coordinates on the real plane as

$$H_\pm = \mp \frac{e^{\mp i\phi}}{q - q^{-1}} \left\{ \left(\rho + \frac{1}{\rho} \right) q^{\rho\partial_\rho \mp (N/2)} - \rho q^{\mp i\partial_\phi \pm (3N/2)} - \frac{1}{\rho} q^{\pm i\partial_\phi \mp (N/2)} \right\}. \quad (2.8)$$

Basis functions $\Psi_{MNq}^J(z, \bar{z})$ for the $(2J+1)$ -dimensional irrep of $\mathfrak{su}_q(2)$ satisfy the relations [4]

$$\begin{aligned} H_3 \Psi_{MNq}^J &= M \Psi_{MNq}^J, & H_\pm \Psi_{MNq}^J &= ([J \mp M]_q [J \pm M + 1]_q)^{1/2} \Psi_{M\pm 1, Nq}^J, \\ \mathcal{C} \Psi_{MNq}^J &= [J]_q [J + 1]_q \Psi_{MNq}^J, & M &= \{-J, -J + 1, \dots, J\}, \quad |N| \leq J, \end{aligned} \quad (2.9)$$

where J , M and N are simultaneously integers or half-integers. Let us remark that, when $q \in S^1$, the existence of such a representation implies that the factorials do not vanish, hence that q is not a root of unity.

Following Rideau and Winternitz [6], let us write $\Psi_{MNq}^J(z, \bar{z})$ as

$$\Psi_{MNq}^J(z, \bar{z}) = N_{MNq}^J Q_{Jq}(\eta) q^{-NM/2} R_{MNq}^J(\eta) \bar{z}^{M+N}, \quad \eta = z\bar{z}. \quad (2.10)$$

Here, N_{MNq}^J is a constant, which can be expressed as

$$\begin{aligned} N_{MNq}^J &= C_{JNq} \left(\frac{[J+M]_q!}{[J-M]_q! [2J]_q!} \right)^{1/2}, \\ C_{JNq} &= \frac{1}{\sqrt{2\pi}} \left(\frac{[J+N]_q! [2J+1]_q!}{[J-N]_q!} \right)^{1/2} \gamma(J, N, q), \end{aligned} \quad (2.11)$$

in terms of some yet undetermined normalization constant $\gamma(J, N, q)$, and q -factorials, defined by $[x]_q! \equiv [x]_q [x-1]_q \dots [1]_q$ if $x \in \mathbb{N}^+$, $[0]_q! \equiv 1$, and $([x]_q!)^{-1} \equiv 0$ if $x \in \mathbb{N}^-$. Equation (2.10) also contains two functions of η , $Q_{Jq}(\eta)$ and $R_{MNq}^J(\eta)$. The latter is a polynomial, whose explicit form is given by

$$\begin{aligned} R_{MNq}^J(\eta) &= [J-N]_q! [J-M]_q! \\ &\times \sum_k \frac{(-\eta)^k}{[k]_q! [J-M-k]_q! [J-N-k]_q! [M+N+k]_q!}, \end{aligned} \quad (2.12)$$

the summation over k being restricted by the condition that all the factorials in the denominator be positive. The former is defined by the functional equation

$$Q_{Jq}(q^2\eta)(1+\eta) = Q_{Jq}(\eta)(1+q^{-2J}\eta), \quad (2.13)$$

whose solution, only determined up to an arbitrary multiplicative factor $f_{Jq}(\eta)$ such that

$$f_{Jq}(q^2\eta) = f_{Jq}(\eta), \quad (2.14)$$

will be discussed in detail for both $q \in \mathbb{R}^+$, and generic $q \in S^+$, in the next section.

In terms of spherical coordinates, Eq. (2.10) becomes [6]

$$\begin{aligned} \Psi_{MNq}^J(\theta, \phi) &= C_{JNq} \left(\frac{[J-N]_q!}{[J+N]_q! [2J]_q!} \right)^{1/2} i^{-2J+M+N} q^{-NM/2} \\ &\times P_{MNq}^J(\cos \theta) e^{-i(M+N)\phi}, \end{aligned} \quad (2.15)$$

where

$$\begin{aligned} P_{MNq}^J(\xi) &= i^{2J-M-N} \left(\frac{[J+M]_q! [J+N]_q!}{[J-M]_q! [J-N]_q!} \right)^{1/2} \eta^{(M+N)/2} Q_{Jq}(\eta) R_{MNq}^J(\eta), \\ \xi &= \cos \theta, \quad \eta = \frac{1+\xi}{1-\xi} = \cot^2 \frac{\theta}{2}, \end{aligned} \quad (2.16)$$

are q -Vilenkin functions. For integer J values, the functions $\Psi_{M0q}^J(\theta, \phi)$ are proportional to q -spherical harmonics, while $P_{Jq}(\xi) \equiv P_{M0q}^J(\xi)$ are q -analogues of Legendre polynomials.

In the $q \rightarrow 1$ limit, the $\text{su}_q(2)$ realization (2.6) goes over into the $\text{su}(2)$ realization

$$H_3 = -z\partial_z + \bar{z}\partial_{\bar{z}} - N, \quad H_+ = -\partial_z - \bar{z}^2\partial_{\bar{z}} + N\bar{z}, \quad H_- = z^2\partial_z + \partial_{\bar{z}} + Nz, \quad (2.17)$$

the constant $\gamma(J, N, q)$ into $\gamma(J, N, 1) = 1$, and the q -Vilenkin functions into ordinary ones $P_{MN}^J(\xi)$. The latter are given by Eq. (2.16), where $[x]_q \rightarrow x$, and $Q_{Jq}(\eta) \rightarrow Q_J(\eta) = (1 + \eta)^{-J}$. The operators (2.17) satisfy Eq. (2.4), and the functions Ψ_{MN}^J , $J = |N|, |N| + 1, \dots$, $M = -J, -J + 1, \dots, J$, form an orthonormal set with respect to the scalar product

$$\langle \psi_1 | \psi_2 \rangle = 2 \int \frac{dz d\bar{z}}{(1 + z\bar{z})^2} \overline{\psi_1(z, \bar{z})} \psi_2(z, \bar{z}) = \frac{1}{2} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \overline{\psi_1(\theta, \phi)} \psi_2(\theta, \phi), \quad (2.18)$$

where the integral over z, \bar{z} extends over the whole complex plane.

III DETERMINATION OF $Q_{Jq}(\eta)$

Following Rideau and Winternitz [6], as a solution of Eq. (2.13), we may consider the function

$$Q_{Jq}(\eta) = {}_1\Phi_0(q^{2J}; -; q^2, -q^{-2J}\eta) = {}_1\Phi_0(q^{-2J}; -; q^{-2}, -q^{-2}\eta), \quad (3.1)$$

where ${}_1\Phi_0$ is a basic hypergeometric series in the notations of Ref. [3].

For $q \in \mathbb{R}^+$, use of the q -binomial theorem [3] leads to the expressions

$$Q_{Jq}(\eta) = \prod_{k=0}^{\infty} \frac{(1 + q^{2k}\eta)}{(1 + q^{-2J+2k}\eta)} \quad (3.2)$$

if $0 < q < 1$, and

$$Q_{Jq}(\eta) = \prod_{k=0}^{\infty} \frac{(1 + q^{-2J-2k-2}\eta)}{(1 + q^{-2k-2}\eta)} \quad (3.3)$$

if $q > 1$. For integer J values, both expressions reduce to the inverse of a polynomial,

$$Q_{Jq}(\eta) = \prod_{k=0}^{J-1} \left(\frac{1}{1 + \eta q^{-2J+2k}} \right), \quad (3.4)$$

whereas for half-integer J values, we are left with convergent infinite products.

For generic $q \in S^1$ and integer J values, Eq. (3.4) still remains a valid solution of Eq. (2.13). However, for half-integer J values, the infinite products contained in Eqs. (3.2)

and (3.3), as well as other expressions of ${}_1\Phi_0$ in terms of infinite series or products, found in Refs. [1, 3], are divergent. We have therefore to look for another solution to Eq. (2.13).

For such a purpose, let us linearize Eq. (2.13) into

$$K_{Jq}(q^2\eta) - K_{Jq}(\eta) = \ln \frac{1 + q^{-2J}\eta}{1 + \eta}, \quad (3.5)$$

by setting

$$K_{Jq}(\eta) = \ln Q_{Jq}(\eta). \quad (3.6)$$

In terms of the operator $X \equiv \eta \partial_\eta$, Eq. (3.5) can be rewritten as

$$(q^{2X} - 1) K_{Jq}(\eta) = (q^{-2JX} - 1) \ln(1 + \eta). \quad (3.7)$$

Let us consider the difference equation

$$(q^X - q^{-X}) L_q(\eta) \equiv L_q(q\eta) - L_q(q^{-1}\eta) = \ln(1 + \eta). \quad (3.8)$$

If we are able to find a solution to the latter, then

$$K_{Jq}(\eta) \equiv q^{-X} (q^{-2JX} - 1) L_q(\eta) = L_q(q^{-2J-1}\eta) - L_q(q^{-1}\eta) \quad (3.9)$$

will be a solution of Eq. (3.7).

We will now proceed to demonstrate

Lemma III.1 *For $0 < \eta < \infty$, and $q = e^{i\tau}$ different from a root of unity, the function*

$$L_q(\eta) = \frac{1}{2\pi i} \int_0^\infty \frac{dt}{t(1+t)} \ln(1 + \eta t^{\tau/\pi}), \quad \text{if } 0 < \tau < \pi, \quad (3.10)$$

$$L_q(\eta) = -\frac{1}{2\pi i} \int_0^\infty \frac{dt}{t(1+t)} \ln(1 + \eta t^{-\tau/\pi}), \quad \text{if } -\pi < \tau < 0, \quad (3.11)$$

is a solution of Eq. (3.8).

Proof. We note that if some function $L_q(\eta)$ is a solution of Eq. (3.8) for $q = e^{i\tau}$, $0 < \tau < \pi$, then $-L_{q^{-1}}(\eta)$ is also a solution of the same. Hence, Eq. (3.11) directly results from Eq. (3.10). It is also a simple matter to show that the integral on the right-hand side of Eq. (3.10) is convergent. It therefore only remains to prove that the latter satisfies

Eq. (3.8). For such a purpose, we have to separately consider the integral when η is replaced by $\eta e^{i\tau}$, or by $\eta e^{-i\tau}$.

Let us introduce a function $M(v)$ of a complex variable v , defined by

$$M(v) = [v(1+v)]^{-1} \ln \left(1 + \eta e^{-i\tau} v^{\tau/\pi} \right), \quad (3.12)$$

where on the right-hand side, there appear two multivalued functions $v^{\tau/\pi}$, and $\ln w$, where $w = 1 + \eta e^{-i\tau} v^{\tau/\pi}$.

For the function $v^{\tau/\pi}$, let us choose a branch cut along the positive real axis, so that $v^{\tau/\pi} = |v|^{\tau/\pi} \exp(i\tau\alpha/\pi)$, where $v = |v| \exp(i\alpha)$, and $0 < \alpha < 2\pi$. On the two sides of such a cut, the argument of the logarithm on the right-hand side of Eq. (3.12), takes the values

$$w_+ = 1 + \eta e^{-i\tau} |v|^{\tau/\pi} \quad \text{if } \operatorname{Re} v > 0, \operatorname{Im} v = +0, \quad (3.13)$$

and

$$w_- = 1 + \eta e^{i\tau} |v|^{\tau/\pi} \quad \text{if } \operatorname{Re} v > 0, \operatorname{Im} v = -0, \quad (3.14)$$

respectively.

Considering next the function $\ln w$, it is easy to show that its branch point at $w = 0$, and its branch cut along the negative real axis in the complex w plane cannot be reached within the truncated v plane, since the condition $\exp[i\tau(\alpha/\pi - 1)] = -1$ cannot be fulfilled for $0 < \tau < \pi$, and $0 < \alpha < 2\pi$.

Hence, when integrating the function $M(v)$ in the complex v plane, one should consider contours avoiding the branch point $v = 0$, the branch cut $\operatorname{Re} v > 0, \operatorname{Im} v = 0$, and the simple pole at $v = -1$. Let us consider the two vanishing contour integrals

$$\int_{\Gamma^+} M(v) dv = \int_{\Gamma^-} M(v) dv = 0, \quad (3.15)$$

where Γ^+ and Γ^- are the paths in the upper and lower halves of the v plane, displayed on Fig. 1. The former consists of the upper half C_A^+ of a large circle of radius A centred at the origin, and described in the counterclockwise sense, the upper halves C_a^+ , $C_{a'}^+$ of two small circles of radius a , a' , centred at $v = 0$ and $v = -1$, respectively, both described in the clockwise sense, and three straight lines L_1^+ , L_2^+ , L_3^+ lying just above the real axis, and

going from $-A$ to $-1 - a'$, from $-1 + a'$ to $-a$, and from a to A , respectively. The latter path Γ^- is defined in a similar way.

Taking now Eqs. (3.13) and (3.14) into account, we obtain

$$2\pi i \left(L_q(q\eta) - L_q(q^{-1}\eta) \right) = \lim_{\substack{a \rightarrow 0 \\ A \rightarrow \infty}} \left\{ \int_{L_3^-} M(v) dv - \int_{L_3^+} M(v) dv \right\}. \quad (3.16)$$

Owing to Eq. (3.15), each of the integrals on the right-hand side of Eq. (3.16) can be rewritten in terms of integrals along the other parts of the path Γ^- or Γ^+ . Those along L_1^+ (resp. L_2^+) and L_1^- (resp. L_2^-) obviously cancel. Furthermore

$$\lim_{A \rightarrow \infty} \left| \int_{C_A^+} M(v) dv - \int_{C_A^-} M(v) dv \right| \sim \lim_{A \rightarrow \infty} \frac{\ln A}{A} = 0, \quad (3.17)$$

and

$$\lim_{a \rightarrow 0} \left| \int_{C_a^+} M(v) dv - \int_{C_a^-} M(v) dv \right| \sim \lim_{a \rightarrow 0} a^{\tau/\pi} = 0, \quad (3.18)$$

so that

$$2\pi i \left(L_q(q\eta) - L_q(q^{-1}\eta) \right) = - \int_{C_{a'}'} M(v) dv = -2\pi i \operatorname{Res} M(-1) = 2\pi i \ln(1 + \eta), \quad (3.19)$$

where $C_{a'}'$ denotes the circle of radius a' centred at $v = -1$, and described in the counter-clockwise sense. Equation (3.19) completes the proof. \blacksquare

The results of the present section can be collected into

Proposition III.2 *The function $Q_{Jq}(\eta)$, appearing on the right-hand side of Eq. (2.10), is given by Eq. (3.4) for integer J values, and either $q \in \mathbb{R}^+$ or generic $q \in S^1$, and by Eqs. (3.2) and (3.3) for half-integer J values, and $q \in \mathbb{R}^+$. For half-integer J values, and generic $q \in S^1$, it can be expressed as*

$$Q_{Jq}(\eta) = \exp \left\{ L_q \left(q^{-2J-1}\eta \right) - L_q \left(q^{-1}\eta \right) \right\}, \quad (3.20)$$

where $L_q(\eta)$ admits the integral representation given in Lemma III.1.

IV UNITARIZATION OF THE REPRESENTATIONS OF $\mathfrak{su}_q(2)$ ON S^2

In the present section, we will determine a new scalar product $\langle \psi_1 | \psi_2 \rangle_q$ that unitarizes the realization (2.6) of $\mathfrak{su}_q(2)$, and goes over into the old one $\langle \psi_1 | \psi_2 \rangle$, defined in Eq. (2.18), whenever $q \rightarrow 1$. For such a purpose, we shall first impose that Eq. (2.4) is satisfied by the realization (2.6) with respect to $\langle \psi_1 | \psi_2 \rangle_q$. The residual arbitrariness in the measure will then be lifted by demanding that $\langle \psi_1 | \psi_2 \rangle_q$ satisfies the usual properties of a scalar product.

We shall successively consider hereunder the cases where $q \in \mathbb{R}^+$, and generic $q \in S^1$.

A The case where $q \in \mathbb{R}^+$

Let us make the following ansatz for $\langle \psi_1 | \psi_2 \rangle_q$,

$$\begin{aligned} \langle \psi_1 | \psi_2 \rangle_q = & \int_0^\infty d\rho \int_0^{2\pi} d\phi \left(\overline{A_q \psi_1(\rho, \phi, q)} f_1(\rho, q) q^{a_1 \rho \partial_\rho} \psi_2(\rho, \phi, q) \right. \\ & \left. + \overline{\psi_1(\rho, \phi, q)} f_2(\rho, q) q^{a_2 \rho \partial_\rho} A_q \psi_2(\rho, \phi, q) \right), \end{aligned} \quad (4.1)$$

in terms of the polar coordinates ρ, ϕ on the real plane, defined in Eq. (2.1). Here a_1, a_2 , and $f_1(\rho, q), f_2(\rho, q)$ are some yet undetermined constants and functions of the indicated arguments, respectively, and $A_q \equiv q^{-2q\partial_q}$ is the operator that changes q into q^{-1} , when acting on any function of q ,

$$A_q \psi(\rho, \phi, q) = \psi(\rho, \phi, q^{-1}). \quad (4.2)$$

It is easy to check that

$$\langle \psi_1 | H_3 \psi_2 \rangle_q = \langle H_3 \psi_1 | \psi_2 \rangle_q \quad (4.3)$$

with respect to (4.1). Let us now impose the condition

$$\langle \psi_1 | H_+ \psi_2 \rangle_q = \langle H_- \psi_1 | \psi_2 \rangle_q. \quad (4.4)$$

By combining Eqs. (2.6) and (4.1), the left-hand side of this condition can be written as

$$\langle \psi_1 | H_+ \psi_2 \rangle_q = (q - q^{-1})^{-1} \int_0^\infty d\rho \int_0^{2\pi} d\phi$$

$$\begin{aligned}
& \times \left\{ \overline{\psi_1(\rho, \phi, q^{-1})} f_1(\rho, q) e^{-i\phi} \left(- \left(q^{a_1} \rho + \frac{1}{q^{a_1} \rho} \right) q^{\rho \partial_\rho - (N/2)} \right. \right. \\
& \quad \left. \left. + q^{a_1} \rho q^{-i\partial_\phi + (3N/2)} + \frac{1}{q^{a_1} \rho} q^{i\partial_\phi - (N/2)} \right) \psi_2(q^{a_1} \rho, \phi, q) \right. \\
& \quad \left. - \overline{\psi_1(\rho, \phi, q)} f_2(\rho, q) e^{-i\phi} \left(- \left(q^{a_2} \rho + \frac{1}{q^{a_2} \rho} \right) q^{-\rho \partial_\rho + (N/2)} \right. \right. \\
& \quad \left. \left. + q^{a_2} \rho q^{i\partial_\phi - (3N/2)} + \frac{1}{q^{a_2} \rho} q^{-i\partial_\phi + (N/2)} \right) \psi_2(q^{a_2} \rho, \phi, q^{-1}) \right\}. \tag{4.5}
\end{aligned}$$

After integrating by parts and making some straightforward transformations, it becomes

$$\begin{aligned}
\langle \psi_1 | H_+ \psi_2 \rangle_q &= (q - q^{-1})^{-1} \int_0^\infty d\rho \int_0^{2\pi} d\phi e^{-i\phi} \\
& \times \left\{ - \left(\left(q^{a_1-1} \rho + \frac{1}{q^{a_1-1} \rho} \right) f_1(q^{-1} \rho, q) q^{-\rho \partial_\rho - 1 - (N/2)} \overline{\psi_1(\rho, \phi, q^{-1})} \right) \psi_2(q^{a_1} \rho, \phi, q) \right. \\
& + \left(\left(\rho q^{i\partial_\phi + a_1 + 1 + (3N/2)} + \frac{1}{\rho} q^{-i\partial_\phi - a_1 - 1 - (N/2)} \right) \overline{\psi_1(\rho, \phi, q^{-1})} \right) f_1(\rho, q) \psi_2(q^{a_1} \rho, \phi, q) \\
& + \left(\left(q^{a_2+1} \rho + \frac{1}{q^{a_2+1} \rho} \right) f_2(q\rho, q) q^{\rho \partial_\rho + 1 + (N/2)} \overline{\psi_1(\rho, \phi, q)} \right) \psi_2(q^{a_2} \rho, \phi, q^{-1}) \\
& - \left(\left(\rho q^{-i\partial_\phi + a_2 - 1 - (3N/2)} + \frac{1}{\rho} q^{i\partial_\phi - a_2 + 1 + (N/2)} \right) \overline{\psi_1(\rho, \phi, q)} \right) f_2(\rho, q) \\
& \left. \times \psi_2(q^{a_2} \rho, \phi, q^{-1}) \right\}. \tag{4.6}
\end{aligned}$$

On the other hand, for real q values the right-hand side of Eq. (4.4) can be written as

$$\begin{aligned}
\langle H_- \psi_1 | \psi_2 \rangle_q &= (q - q^{-1})^{-1} \int_0^\infty d\rho \int_0^{2\pi} d\phi e^{-i\phi} \\
& \times \left\{ \left(\left\{ - \left(\rho + \frac{1}{\rho} \right) q^{-\rho \partial_\rho - (N/2)} + \rho q^{i\partial_\phi + (3N/2)} + \frac{1}{\rho} q^{-i\partial_\phi - (N/2)} \right\} \overline{\psi_1(\rho, \phi, q^{-1})} \right) \right. \\
& \times f_1(\rho, q) \psi_2(q^{a_2} \rho, \phi, q) \\
& + \left(\left\{ \left(\rho + \frac{1}{\rho} \right) q^{\rho \partial_\rho + (N/2)} - \rho q^{-i\partial_\phi - (3N/2)} - \frac{1}{\rho} q^{i\partial_\phi + (N/2)} \right\} \overline{\psi_1(\rho, \phi, q)} \right) \\
& \left. \times f_2(\rho, q) \psi_2(q^{a_2} \rho, \phi, q^{-1}) \right\}. \tag{4.7}
\end{aligned}$$

It now remains to equate the right-hand side of Eq. (4.6) with that of Eq. (4.7). Both of them being some linear combinations of four different types of terms, containing one of the operators $q^{-i\partial_\phi}$, $q^{i\partial_\phi}$, $q^{-\rho \partial_\rho}$, or $q^{\rho \partial_\rho}$, acting on some function, respectively, it is sufficient

to separately equate such terms. The conditions on the first two classes of terms impose that

$$a_1 = -1, \quad a_2 = 1, \quad (4.8)$$

while those on the last two lead to the equations

$$\begin{aligned} q^{-1} \left(q^{-2} \rho + \frac{1}{q^{-2} \rho} \right) f_1(q^{-1} \rho, q) &= \left(\rho + \frac{1}{\rho} \right) f_1(\rho, q), \\ q \left(q^2 \rho + \frac{1}{q^2 \rho} \right) f_2(q \rho, q) &= \left(\rho + \frac{1}{\rho} \right) f_2(\rho, q), \end{aligned} \quad (4.9)$$

whose solutions are given by

$$f_1(\rho, q) = \frac{B_1(q) q^{-1} \rho}{(1 + \rho^2)(1 + q^{-2} \rho^2)}, \quad f_2(\rho, q) = \frac{B_2(q) q \rho}{(1 + \rho^2)(1 + q^2 \rho^2)}, \quad (4.10)$$

in terms of two undetermined constants $B_1(q)$, and $B_2(q)$.

Let us now further restrict the sesquilinear form (4.1), where substitutions (4.8) and (4.10) have been made, by imposing that it is Hermitian, i.e.,

$$\overline{\langle \psi_1 | \psi_2 \rangle_q} = \langle \psi_2 | \psi_1 \rangle_q. \quad (4.11)$$

By a straightforward calculation, similar to that carried out for condition (4.4), it can be shown that Eq. (4.11) leads to the relation

$$B_2(q) = \overline{B_1(q)}. \quad (4.12)$$

As a consequence, there only remains a single undetermined constant $B(q) \equiv B_1(q)$ in Eq. (4.1). At this stage, it is important to notice that had we only considered a single term, instead of two, in Eq. (4.1), it would have been impossible to fulfil condition (4.11).

In addition, we remark that Eqs. (4.4) and (4.11) imply that

$$\langle \psi_1 | H_- \psi_2 \rangle_q = \langle H_+ \psi_1 | \psi_2 \rangle_q. \quad (4.13)$$

Hence, all the Hermiticity conditions (2.4) on the $\text{su}_q(2)$ generators are satisfied by the form defined in Eqs. (4.1), (4.8), (4.10), and (4.12). The functions $\Psi_{MNq}^J(z, \bar{z})$, defined in Eq. (2.10), and corresponding to a fixed N value, but different J and/or M values, are therefore orthogonal with respect to such a form.

To make $\langle \psi_1 | \psi_2 \rangle$ into a scalar product, it only remains to impose that it is a positive definite form. Since we also want that in the resulting Hilbert space, the functions Ψ_{MNq}^J with given J and N values, and $M = -J, -J+1, \dots, J$, form an orthonormal basis for the $\text{su}_q(2)$ irrep characterized by J , a condition that combines both requirements is

$$\langle \Psi_{MNq}^J | \Psi_{MNq}^J \rangle = 1, \quad M = -J, -J+1, \dots, J. \quad (4.14)$$

By using Eqs. (2.5) and (2.9) for $M \neq J$, Eq. (4.14) can be transformed into the condition

$$\langle \Psi_{JNq}^J | \Psi_{JNq}^J \rangle = 1. \quad (4.15)$$

In Appendix A, the squared norm of Ψ_{JNq}^J is calculated by using Eqs. (2.10), (2.11), (2.12), (3.2), (3.3), and by taking Eqs. (4.1), (4.8), (4.10), and (4.12) into account. The resulting condition (4.15) reads

$$\frac{\ln q}{q - q^{-1}} \left(B(q) \overline{\gamma(J, N, q^{-1})} \gamma(J, N, q) + \overline{B(q)} \overline{\gamma(J, N, q)} \gamma(J, N, q^{-1}) \right) = 1. \quad (4.16)$$

Since in the limit $q \rightarrow 1$, $\gamma(J, N, q) \rightarrow 1$, we may choose

$$\gamma(J, N, q) = 1, \quad B(q) = \overline{B(q)} = \frac{q - q^{-1}}{2 \ln q} = \frac{\sinh \tau}{\tau}. \quad (4.17)$$

For $q \rightarrow 1$ or $\tau \rightarrow 0$, we find that $B(q) \rightarrow 1$, so that $\langle \psi_1 | \psi_2 \rangle_q \rightarrow \langle \psi_1 | \psi_2 \rangle$, where the latter is given by Eq. (2.18), as it should be.

The results obtained can be summarized as follows:

Proposition IV.1 *For $q \in \mathbb{R}^+$, the scalar product*

$$\begin{aligned} \langle \psi_1 | \psi_2 \rangle_q &= \frac{q - q^{-1}}{2 \ln q} \int dz d\bar{z} \left(\overline{\psi_1(z, \bar{z}, q^{-1})} \frac{1}{(1 + z\bar{z})(1 + q^{-2}z\bar{z})} q^{-z\partial_z - \bar{z}\partial_{\bar{z}} - 1} \psi_2(z, \bar{z}, q) \right. \\ &\quad \left. + \overline{\psi_1(z, \bar{z}, q)} \frac{1}{(1 + z\bar{z})(1 + q^2z\bar{z})} q^{z\partial_z + \bar{z}\partial_{\bar{z}} + 1} \psi_2(z, \bar{z}, q^{-1}) \right), \end{aligned} \quad (4.18)$$

or

$$\begin{aligned} \langle \psi_1 | \psi_2 \rangle_q &= \frac{q - q^{-1}}{8 \ln q} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \\ &\times \left(\overline{\psi_1(\theta, \phi, q^{-1})} \frac{1}{\sin^2(\theta/2) + q^{-2} \cos^2(\theta/2)} q^{\sin \theta \partial_\theta - 1} \psi_2(\theta, \phi, q) \right. \\ &\quad \left. + \overline{\psi_1(\theta, \phi, q)} \frac{1}{\sin^2(\theta/2) + q^2 \cos^2(\theta/2)} q^{-\sin \theta \partial_\theta + 1} \psi_2(\theta, \phi, q^{-1}) \right), \end{aligned} \quad (4.19)$$

unitarizes the $su_q(2)$ realization (2.6), where N may take any integer or half-integer value. The functions $\Psi_{MNq}^J(z, \bar{z})$, or $\Psi_{MNq}^J(\theta, \phi)$, defined in Eqs. (2.10) and (2.15), where $J = |N|, |N| + 1, \dots, M = -J, -J + 1, \dots, J$, and $\gamma(J, N, q) = 1$, form an orthonormal set with respect to such a scalar product.

From Proposition IV.1, we easily obtain

Corollary IV.2 For $q \in \mathbb{R}^+$, the q -Vilenkin functions $P_{MNq}^J(\xi)$, defined in Eq. (2.16), satisfy the orthonormality relation

$$\begin{aligned} \frac{q - q^{-1}}{4 \ln q} \int_{-1}^{+1} d\xi \left(\overline{P_{MNq^{-1}}^{J'}(\xi)} \frac{1}{q + q^{-1} - (q - q^{-1})\xi} q^{(\xi^2 - 1)\partial_\xi} P_{MNq}^J(\xi) \right. \\ \left. + \overline{P_{MNq}^{J'}(\xi)} \frac{1}{q + q^{-1} + (q - q^{-1})\xi} q^{-(\xi^2 - 1)\partial_\xi} P_{MNq^{-1}}^J(\xi) \right) = \frac{\delta_{J', J}}{[2J + 1]_q}. \end{aligned} \quad (4.20)$$

B The case where $q \in S^1$

Whenever $q \in S^1$, the ansatz (4.1) does not work, because though Eq. (4.6) remains valid, Eq. (4.7) is changed in such a way that both cannot be matched. Let us therefore change Eq. (4.1) into the following ansatz

$$\begin{aligned} \langle \psi_1 | \psi_2 \rangle_q = & \int_0^\infty d\rho \int_0^{2\pi} d\phi \left(\overline{\psi_1(\rho, \phi, q)} f_1(\rho, q) q^{a_1 \rho \partial_\rho} \psi_2(\rho, \phi, q) \right. \\ & \left. + \overline{A_q \psi_1(\rho, \phi, q)} f_2(\rho, q) q^{a_2 \rho \partial_\rho} A_q \psi_2(\rho, \phi, q) \right), \end{aligned} \quad (4.21)$$

where $a_1, a_2, f_1(\rho, q), f_2(\rho, q)$, and A_q keep the same meaning as before.

Condition (4.3) is again automatically satisfied. Turning now to condition (4.4), it is easy to see that Eqs. (4.6) and (4.7) remain valid, except for the interchange of $\overline{\psi_1(\rho, \phi, q)}$ with $\overline{\psi_1(\rho, \phi, q^{-1})}$. Hence, Eq. (4.4) is also fulfilled by choosing $a_1, a_2, f_1(\rho, q)$, and $f_2(\rho, q)$ as given in Eqs. (4.8), and (4.10).

A difference with the case where $q \in \mathbb{R}^+$ appears when imposing the Hermiticity condition (4.11). The latter is now equivalent to the relations

$$\overline{B_1(q)} = B_1(q), \quad \overline{B_2(q)} = B_2(q), \quad (4.22)$$

showing that the real constants $B_1(q)$, and $B_2(q)$ remain independent. In the present case, keeping only one of the two terms on the right-hand side of Eq. (4.21) would therefore lead to a well-behaved scalar product.

As shown in Appendix B, condition (4.15) now reads

$$\frac{\ln q}{q - q^{-1}} \left(B_1(q) |\gamma(J, N, q)|^2 + B_2(q) |\gamma(J, N, q^{-1})|^2 \right) = 1. \quad (4.23)$$

Among the infinitely many solutions of this equation, we may select the most symmetrical one,

$$\gamma(J, N, q) = 1, \quad B_1(q) = B_2(q) = \frac{q - q^{-1}}{2 \ln q} = \frac{\sin \tau}{\tau}. \quad (4.24)$$

Hence, whenever $q \rightarrow 1$ or $\tau \rightarrow 0$, the limit of $\langle \psi_1 | \psi_2 \rangle_q$ is again $\langle \psi_1 | \psi_2 \rangle$, as it should be.

In conclusion, we obtain

Proposition IV.3 *For generic $q \in S^1$, the scalar product*

$$\begin{aligned} \langle \psi_1 | \psi_2 \rangle_q &= \frac{q - q^{-1}}{2 \ln q} \int dz d\bar{z} \left(\overline{\psi_1(z, \bar{z}, q)} \frac{1}{(1 + z\bar{z})(1 + q^{-2}z\bar{z})} q^{-z\partial_z - \bar{z}\partial_{\bar{z}} - 1} \psi_2(z, \bar{z}, q) \right. \\ &\quad \left. + \overline{\psi_1(z, \bar{z}, q^{-1})} \frac{1}{(1 + z\bar{z})(1 + q^2z\bar{z})} q^{z\partial_z + \bar{z}\partial_{\bar{z}} + 1} \psi_2(z, \bar{z}, q^{-1}) \right), \end{aligned} \quad (4.25)$$

or

$$\begin{aligned} \langle \psi_1 | \psi_2 \rangle_q &= \frac{q - q^{-1}}{8 \ln q} \int_0^\pi d\theta \sin \theta \int_0^{2\pi} d\phi \\ &\quad \times \left(\overline{\psi_1(\theta, \phi, q)} \frac{1}{\sin^2(\theta/2) + q^{-2} \cos^2(\theta/2)} q^{\sin \theta \partial_\theta - 1} \psi_2(\theta, \phi, q) \right. \\ &\quad \left. + \overline{\psi_1(\theta, \phi, q^{-1})} \frac{1}{\sin^2(\theta/2) + q^2 \cos^2(\theta/2)} q^{-\sin \theta \partial_\theta + 1} \psi_2(\theta, \phi, q^{-1}) \right), \end{aligned} \quad (4.26)$$

unitarizes the $su_q(2)$ realization (2.6), where N may take any integer or half-integer value. The functions $\Psi_{MNq}^J(z, \bar{z})$, or $\Psi_{MNq}^J(\theta, \phi)$, defined in Eqs. (2.10) and (2.15), where $J = |N|, |N| + 1, \dots, M = -J, -J + 1, \dots, J$, and $\gamma(J, N, q) = 1$, form an orthonormal set with respect to such a scalar product.

Corollary IV.4 *For generic $q \in S^1$, the q -Vilenkin functions $P_{MNq}^J(\xi)$, defined in Eq. (2.16), satisfy the orthonormality relation*

$$\begin{aligned} \frac{q - q^{-1}}{4 \ln q} \int_{-1}^{+1} d\xi \left(\overline{P_{MNq}^{J'}(\xi)} \frac{1}{q + q^{-1} - (q - q^{-1})\xi} q^{(\xi^2 - 1)\partial_\xi} P_{MNq}^J(\xi) \right. \\ \left. + \overline{P_{MNq^{-1}}^{J'}(\xi)} \frac{1}{q + q^{-1} + (q - q^{-1})\xi} q^{-(\xi^2 - 1)\partial_\xi} P_{MNq^{-1}}^J(\xi) \right) = \frac{\delta_{J', J}}{[2J + 1]_q}. \end{aligned} \quad (4.27)$$

V Conclusion

In the present paper, we did extend the study of the $\mathrm{su}_q(2)$ representations on a real two-dimensional sphere, carried out by Rideau and Winternitz [6], in two ways.

Firstly, we did prove that such representations exist not only for $q \in \mathbb{R}^+$, but also for generic $q \in S^1$. For such a purpose, we did provide an integral representation for the functions $Q_{Jq}(\eta)$, entering the definition of the q -Vilenkin functions, whenever J takes any half-integer value.

Secondly, we did unitarize the representations by determining appropriate scalar products for both ranges of q values. Such scalar products are expressed in terms of ordinary integrals, instead of q -integrals, as is usually the case [5].

The resulting orthonormality relations for the q -Vilenkin and related functions should play an important role in applications to quantum mechanics, such as those considered in Refs. [10, 11].

APPENDIX A: PROOF OF EQUATION (4.16)

The purpose of this appendix is to evaluate the squared norm of the function $\Psi_{JNq}^J(z, \bar{z})$ when the scalar product (4.1) is used, and Eqs. (4.8), (4.10), and (4.12) are taken into account.

From Eqs. (2.2), (2.10), (2.11), and (2.12), Ψ_{JNq}^J can be written in polar coordinates as

$$\Psi_{JNq}^J = \frac{C_{JNq}}{[J+N]_q!} q^{-JN/2} Q_{Jq}(\rho^2) \rho^{J+N} e^{-i(J+N)\phi}. \quad (\text{A1})$$

Its squared norm can therefore be expressed as

$$\begin{aligned} \langle \Psi_{JNq}^J | \Psi_{JNq}^J \rangle_q &= \frac{\pi}{([J+N]_q!)^2} \left(B(q) \overline{C_{JNq^{-1}}} C_{JNq} q^{-J-N-1} \mathcal{I}_q \right. \\ &\quad \left. + \overline{B(q)} \overline{C_{JNq}} C_{JNq^{-1}} q^{J+N+1} \mathcal{I}_{q^{-1}} \right), \end{aligned} \quad (\text{A2})$$

in terms of the integral

$$\mathcal{I}_q = \int_0^\infty d\eta Q_{Jq^{-1}}(\eta) \frac{\eta^{J+N}}{(1+\eta)(1+q^{-2}\eta)} Q_{Jq}(q^{-2}\eta), \quad (\text{A3})$$

and the same with q replaced by q^{-1} .

By introducing Eqs. (3.2) and (3.3) into Eq. (A3), we obtain

$$\mathcal{I}_q = \int_0^\infty d\eta \eta^{J+N} \prod_{k=0}^\infty \frac{(1+q^{2J+2k+2}\eta)}{(1+q^{-2J+2k-2}\eta)} = q^{2(J+1)(J+N+1)} \tilde{B}_{q^2}(J+N+1, J-N+1) \quad (\text{A4})$$

if $0 < q < 1$, and

$$\mathcal{I}_q = \int_0^\infty d\eta \eta^{J+N} \prod_{k=0}^\infty \frac{(1+q^{-2J-2k-4}\eta)}{(1+q^{2J-2k}\eta)} = q^{-2J(J+N+1)} \tilde{B}_{q^{-2}}(J+N+1, J-N+1) \quad (\text{A5})$$

if $q > 1$. In Eqs. (A4) and (A5), we denote by $\tilde{B}_q(x, y)$ Ramanujan's continuous q -analogue of the beta integral [2]

$$\tilde{B}_q(x, y) = \int_0^\infty dt t^{x-1} \prod_{k=0}^\infty \frac{(1+q^{x+y+k}t)}{(1+q^k t)}, \quad 0 < q < 1, \quad (\text{A6})$$

to distinguish it from the discrete q -analogue of the same, known as $B_q(x, y)$ (see e.g. Eq. (1.11.7) of Ref. [3]).

From Eq. (5.8) of Ref. [2], $\tilde{B}_q(x, y)$ is given for generic x values by

$$\tilde{B}_q(x, y) = \frac{\pi}{\sin \pi x} \prod_{k=1}^{\infty} \frac{(1 - q^{k-x})(1 - q^{x+y+k-1})}{(1 - q^k)(1 - q^{y+k-1})}. \quad (\text{A7})$$

The values of x , which appear in Eqs. (A4) and (A5), being $x = J + N + 1 \in \mathbb{N}^+$, we have to calculate the limit of the right-hand side of Eq. (A7) when $x \rightarrow m \in \mathbb{N}^+$. Using L'Hospital rule, we find

$$\lim_{x \rightarrow m} \frac{1 - q^{m-x}}{\sin \pi x} = (-1)^m \frac{\ln q}{\pi}, \quad m \in \mathbb{N}^+. \quad (\text{A8})$$

Hence, for $x = m$, $y = n$, $m, n \in \mathbb{N}^+$, Eq. (A7) becomes

$$\begin{aligned} \tilde{B}_q(m, n) &= (-1)^m (\ln q) \frac{\prod_{k=1}^{m-1} (1 - q^{k-m})}{\prod_{k=1}^m (1 - q^{n+k-1})} \\ &= \frac{(\ln q) q^{-m(n+m-1)/2} [m-1]_{q^{1/2}}! [n-1]_{q^{1/2}}!}{(q^{1/2} - q^{-1/2}) [n+m-1]_{q^{1/2}}!}, \end{aligned} \quad (\text{A9})$$

where in the last step, we introduced q -factorials, defined as in Sec. II.

From Eqs. (A4), (A5), and (A9), it follows that for any $q \in \mathbb{R}^+$

$$\mathcal{I}_q = \frac{2(\ln q) q^{J+N+1} [J+N]_q! [J-N]_q!}{(q - q^{-1}) [2J+1]_q!}. \quad (\text{A10})$$

By taking Eq. (2.11) into account, the squared norm of Ψ_{JNq}^J , defined in Eq. (A2), therefore becomes

$$\langle \Psi_{JNq}^J | \Psi_{JNq}^J \rangle_q = \frac{\ln q}{q - q^{-1}} \left(B(q) \overline{\gamma(J, N, q^{-1})} \gamma(J, N, q) + \overline{B(q)} \overline{\gamma(J, N, q)} \gamma(J, N, q^{-1}) \right), \quad (\text{A11})$$

which proves Eq. (4.16).

APPENDIX B: PROOF OF EQUATION (4.23)

The purpose of this appendix is to evaluate the squared norm of the function $\Psi_{JNq}^J(z, \bar{z})$ when the scalar product (4.21) is used, and Eqs. (4.8), (4.10), and (4.22) are taken into account.

Since for $q \in S^1$, Ψ_{JNq}^J is still given by Eq. (A1), its squared norm reads

$$\begin{aligned} \langle \Psi_{JNq}^J | \Psi_{JNq}^J \rangle_q &= \frac{\pi}{([J+N]_q!)^2} \left(B_1(q) |C_{JNq}|^2 q^{-J-N-1} \mathcal{I}'_q \right. \\ &\quad \left. + B_2(q) |C_{JNq^{-1}}|^2 q^{J+N+1} \mathcal{I}'_{q^{-1}} \right). \end{aligned} \quad (\text{B1})$$

Here \mathcal{I}'_q denotes the integral

$$\mathcal{I}'_q = \int_0^\infty d\eta F_{Jq}(\eta) \eta^{J+N} \quad (\text{B2})$$

with

$$F_{Jq}(\eta) = \overline{Q_{Jq}(\eta)} \frac{1}{(1+\eta)(1+q^{-2}\eta)} Q_{Jq}(q^{-2}\eta). \quad (\text{B3})$$

According to whether J is integer or half-integer, we have to insert Eq. (3.4) or Eq. (3.20) into Eq. (B3). In both cases, the result reads

$$F_{Jq}(\eta) = \prod_{p=0}^{2J+1} \frac{1}{1+q^{2J-2p}\eta}. \quad (\text{B4})$$

This is obvious in the former case. In the latter, by using the property $\overline{L_q(\eta)} = -L_q(\eta)$, Eq. (B3) can be transformed into

$$\begin{aligned} F_{Jq}(\eta) &= \exp \left\{ -L_q(q^{2J+1}\eta) + L_q(q\eta) \right\} \frac{1}{(1+\eta)(1+q^{-2}\eta)} \exp \left\{ L_q(q^{-2J-3}\eta) - L_q(q^{-3}\eta) \right\} \\ &= \frac{1}{(1+\eta)(1+q^{-2}\eta)} \exp \left\{ - \sum_{p=0}^{2J+1} \left[L_q(q^{2J+1-2p}\eta) - L_q(q^{2J-1-2p}\eta) \right] \right. \\ &\quad \left. + \left[L_q(q\eta) - L_q(q^{-1}\eta) \right] + \left[L_q(q^{-1}\eta) - L_q(q^{-3}\eta) \right] \right\}. \end{aligned} \quad (\text{B5})$$

Repeated use of Eq. (3.8) for various arguments then directly leads to the searched for result (B4).

To evaluate \mathcal{I}'_q for $F_{Jq}(\eta)$ given by Eq. (B4), we cannot use the same method as that employed in Appendix A to calculate \mathcal{I}_q , because in the q -analogue of the beta integral,

given in Eq. (A6), q is assumed real. Let us therefore rewrite the integrand of \mathcal{I}'_q in the form

$$\eta^{J+N} F_{Jq}(\eta) = \sum_{p=0}^{2J+1} \frac{a_p^{(J)}}{\eta + q^{2p-2J}}, \quad (\text{B6})$$

where the coefficient $a_p^{(J)}$ is the residue of $\eta^{J+N} F_{Jq}(\eta)$ at the pole $\eta = -q^{2p-2J}$, i.e.,

$$a_p^{(J)} = (-1)^{J+N} \frac{q^{J+1}}{(q - q^{-1})^{2J+1}} \times \frac{(-1)^p q^{N(2p-2J)}}{[p]_q! [2J - p + 1]_q!}. \quad (\text{B7})$$

Then

$$\begin{aligned} G_{Jq}(\eta) &\equiv \int d\eta F_{Jq}(\eta) \eta^{J+N} \\ &= (-1)^{J+N} \frac{q^{J+1}}{(q - q^{-1})^{2J+1}} \sum_{p=0}^{2J+1} \frac{(-1)^p q^{N(2p-2J)}}{[p]_q! [2J - p + 1]_q!} \ln(\eta + q^{2p-2J}). \end{aligned} \quad (\text{B8})$$

To calculate the values of $G_{Jq}(\eta)$ for $\eta \rightarrow \infty$ and $\eta = 0$, the following identities [1] will be useful:

$$(1; \eta)_q^{2J+1} \equiv \sum_{p=0}^{2J+1} \left[\begin{matrix} 2J+1 \\ p \end{matrix} \right]_q \eta^p = \prod_{p=0}^{2J} (1 + q^{2p-2J} \eta), \quad (\text{B9})$$

$$\frac{d}{d\eta} (1; \eta)_q^{2J+1} = \sum_{p=0}^{2J+1} \left[\begin{matrix} 2J+1 \\ p \end{matrix} \right]_q p \eta^{p-1} = \sum_{p=0}^{2J} q^{2p-2J} \prod_{\substack{r=0 \\ r \neq p}}^{2J} (1 + q^{2r-2J} \eta), \quad (\text{B10})$$

where

$$\left[\begin{matrix} n \\ p \end{matrix} \right]_q \equiv \frac{[n]_q!}{[p]_q! [n-p]_q!} \quad (\text{B11})$$

is a q -binomial coefficient. From Eq. (B9), we obtain

$$(1; -q^{2N})_q^{2J+1} = \sum_{p=0}^{2J+1} (-1)^p \left[\begin{matrix} 2J+1 \\ p \end{matrix} \right]_q q^{2Np} = 0, \quad (\text{B12})$$

because on the right-hand side, the factor $(1 - q^{2p-2J+2N})$ vanishes for $p = J - N$. Similarly, from Eq. (B10), we get

$$\begin{aligned} \left. \frac{d}{d\eta} (1; \eta)_q^{2J+1} \right|_{\eta = -q^{2N}} &= \sum_{p=0}^{2J+1} (-1)^{p-1} p \left[\begin{matrix} 2J+1 \\ p \end{matrix} \right]_q q^{2N(p-1)} \\ &= (-1)^{J+N} (q - q^{-1})^{2J} [J - N]_q! [J + N]_q! q^{N(2J-1)}, \end{aligned} \quad (\text{B13})$$

since on the right-hand side, only the term corresponding to $p = J - N$ leads to a nonvanishing result.

By noting that for $\eta \gg 1$,

$$\ln(\eta + q^{2p-2J}) \simeq \ln(\eta) + \frac{q^{2p-2J}}{\eta} + O\left(\frac{1}{\eta^2}\right), \quad (\text{B14})$$

it directly results from Eq. (B12) that

$$\lim_{\eta \rightarrow \infty} G_{Jq}(\eta) = 0. \quad (\text{B15})$$

Furthermore, from Eqs. (B12) and (B13), we obtain

$$\begin{aligned} G_{Jq}(0) &= \frac{(-1)^{J+N} q^{J+1} \ln q}{(q - q^{-1})^{2J+1}} \sum_{p=0}^{2J+1} \frac{(-1)^p (2p - 2J) q^{N(2p-2J)}}{[p]_q! [2J - p + 1]_q!} \\ &= \frac{2(-1)^{J+N} q^{J+1} \ln q}{[2J + 1]_q! (q - q^{-1})^{2J+1}} \sum_{p=0}^{2J+1} (-1)^p p \begin{bmatrix} 2J + 1 \\ p \end{bmatrix}_q q^{N(2p-2J)} \\ &= -\frac{2[J + N]_q! [J - N]_q! q^{J+N+1} \ln q}{[2J + 1]_q! (q - q^{-1})}. \end{aligned} \quad (\text{B16})$$

By taking Eqs. (B8), (B15), and (B16) into account, we conclude that for generic $q \in S^1$, \mathcal{I}'_q , defined in Eq. (B2), is given by

$$\mathcal{I}'_q = \frac{2(\ln q) q^{J+N+1} [J + N]_q! [J - N]_q!}{(q - q^{-1}) [2J + 1]_q!}. \quad (\text{B17})$$

By combining this result with Eqs. (2.11), (4.15), and (B1), Eq. (4.23) directly follows.

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Figure caption

Fig. 1. Contours in the complex v plane used in the proof of Lemma III.1.